

Incentives for Information Sharing and Outcome Efforts in Networks

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Abstract

In social or organizational networks, it is often observed that different individuals put different levels of production effort depending on their position in the network. One possible reason is reward sharing, which incentivizes particular agents to spend effort in sharing information with others and increasing their productivity. We model the effort level in a network as a strategic decision made by an agent on how much effort to expend on the complementary tasks of information sharing and production. We conduct a game-theoretic analysis of incentive and information sharing in both hierarchical and general influencer-infleece networks. Our particular interest is in understanding how different reward structures in a network influence this decision. We establish the existence of a unique pure-strategy Nash equilibrium in regard to the choice made by each agent, and study the effect of the quality and cost of communication, and the reward sharing on the effort levels at this equilibrium. Our results show that a larger reward share from an infleece incentivizes the influencer to spend more effort, in equilibrium, on communication, capturing a free-riding behavior of well placed agents. We also address the reverse question of designing an optimal reward sharing scheme that achieves the effort profile which maximizes the system output. In this direction, for a number of stylized networks, we study the Price of Anarchy for this output, and the interplay between information and incentive sharing on mitigating the loss in output due to agent self-interest.

1 Introduction

The organization of economic activity as a means for the efficient co-ordination of effort is a cornerstone of economic theory. In networked organizations, agents are responsible for two processes : information flow and productive effort. The primary aspect of networked organization that we study is the effect of direct and indirect rewards, e.g. due to wages and profit sharing, on the decisions of agents to work *vs.* invest effort in explaining tasks to others. That is, we are interested in the trade-off an agent faces between the *complementary* tasks of direct effort (or production) and what can be considered information propagation (or communication) effort, which can benefit others. In our model, working on a task brings a direct payoff and is costly, whereas investing effort in explaining a task can improve the productivity of others (depending on the quality of communication in the network). This can in turn generate additional indirect reward for an agent through reward sharing incentives.

We model the network as a directed graph, where the direction represents the direction of information flow or communication between nodes and the rewards are shared in the reverse direction. Of

particular interest to us are directed trees, which represent a hierarchy, and are the most prevalent in organizations and firms. In the first part of the paper, our analysis is focused on hierarchies, and in the second, we generalize our results to arbitrary directed graphs. Our goal is to understand the consequences of various organizational parameters on the outcomes of the effort trade-off decision.

In particular, we are interested in the effects of the quality of *influence process*, communication and magnitude of reward sharing, on the equilibrium decisions of agents with regard to how they split time between work and communication efforts. Different networks, such as social and organizational networks, have different purposes and thus different influence processes. Within firms, organizational networks are often hierarchical and there is a long history on the role of organizational structure on economic efficiency going back to Tichy et al. [20] (on social network analysis within organizations). More recently, Radner [15], Ravasz and Barabási [16], Mookherjee [13] study the role of hierarchies; see Van Alstyne [21] for a survey of different perspectives. There is also a growing interest in crowdsourcing, and relevant here, the ability to generate effective networks for solving challenging problems. Our model also captures some aspects of so called ‘diffusion-based task environments’ where agents become aware of tasks through recruitment [14, 22]. For example, the winner of the 2009 DARPA Red Balloon Challenge adopted an indirect reward scheme where the reward associated with successful completion of subtasks was shared with other agents in the network [14]. At the same time modern massive online social networks and online gaming networks¹ require information and incentive propagation to organize activity. In this paper, we draw attention to the *interaction* between various aspects of network influence, such as profit sharing [9], information exchange [4], and influence in networks.

Motivated by the possibility that this phenomenon of splitting effort into production and communication can be understood as a consequence of the strategic behavior of the participants, we adopt a game theoretic perspective where individual members in a networked organization decide on effort levels motivated by their self interest. Agents are coordinated by incentives, including both direct wages and indirect profit sharing. We construct quantitative models of organizations, that are general enough to capture social and economic networks, but specific enough for us to obtain insightful results. We quantify the effects of reward sharing and communication quality on the performance of work organizations in equilibrium. For stylized networks, we quantify the welfare cost that arises because of self-interested behavior, adopting the Price of Anarchy (PoA) framework. In particular, we study the improvement in PoA that can be obtained through the careful design of indirect rewards within the network.

1.1 Overview and Main Results

In the first and major part of this work, we study hierarchies where the network is a directed tree. Each agent decides how to split its effort between (i) production effort, which results in direct payoff for the agent and indirect reward to other agents on the path from the root to the agent, and (ii) communication effort, which serves to improve the productivity of his descendants on the tree (e.g., explaining the problem to others, conveying insights and the goals of the organization). A natural constraint is imposed on the complementary tasks of production and communication, such that the more effort an agent invests in production the less he can communicate. Investing production effort incurs a cost to an agent, in return for some direct payoff. But committing effort to communication the can improve productivity of descendants, which in turn improves their output, should they decide to invest effort in direct work, and thus give an agent a return on investment through an indirect payoff.

Each agent decides, *based on his position in the hierarchy*, how to split his effort between pro-

¹<http://www.eveonline.com/>

duction and communication, in order to maximize the sum of direct payoff and indirect reward, accounting for the cost of effort. For most of our results we adopt an *exponential productivity* (EP) model, where the quality of communication falls exponentially with effort spent in production with a parameter β . The model has the useful property that a pure-strategy Nash equilibrium always exists (Theorem 1) even though the game is non-concave. In a concave game, the agents’ payoffs are concave in their choices (production efforts), and a pure-strategy Nash equilibrium is guaranteed to exist [18]. We develop tight conditions for the uniqueness of the equilibrium (Theorem 2). In addition, for the EP model of communication, the Nash equilibrium can be computed in time that is quadratic in the number of agents, despite the non-concave nature of the problem, by exploiting the hierarchical structure.

Our next result is that for *balanced hierarchies* and in the EP model, there exists a threshold β^* on communication quality parameter β such that if communication performance is below the threshold (communication is ‘good enough’) then the PoA is equal to 1 *for the optimal reward sharing scheme*, while it can be large otherwise (Theorem 4). For β above this threshold (low quality communication), we give closed-form bounds on the PoA (Theorem 5), which we show are tight in special networks, e.g., single-level hierarchies. Thus, even in simple hierarchies, if the communication is not good enough *or* incentives are not chosen correctly, the PoA can be large. This highlights the importance of the design of reward sharing in organizations accounting for both network structure and communication process.

In the second part, we consider general directed network graphs and establish the existence of a pure-strategy Nash equilibrium and a characterization for when this equilibrium is unique (Theorems 6 and 7). We also provide a geometric interpretation of these conditions in terms of the stability properties of a suitably defined Jacobian matrix (Figure 7). This connection between control-theoretic stability and uniqueness of Nash equilibrium in network games is an interesting property of the model.

For ease of reading, some proofs are deferred to the Appendix.

1.2 Prior Work

The study of effort levels in network games, where an agent’s utility depends on actions of neighboring agents has recently received much attention [7]. For example, Ballester et al. [3] show how the level of activity of a given agent depends on the Bonacich centrality of the agent in the network, for a specific utility structure that results in a concave game. Rogers [17] analyzes the efficiency of equilibria in two specific types of games (i) ‘giving’ and (ii) ‘taking’, where an edge means utility is sent on an edge. A strategic model of effort is discussed in the public goods model of Bramoullé and Kranton [5], where utility is concave in individual agents’ efforts, and the structures of the Nash and stable equilibria are shown. Their model applies to a very specific utility structure where the same benefit of the ‘public good’ is experienced by all the first level neighbors on a graph. In our model, the individual utilities can be asymmetric, and depend on the efforts and reward shares in multiple levels on the graph. Building on these efforts our utility model cleanly separate the effects of two types of influence, that we termed information and incentives.

The recent DARPA Red Balloon Challenge, and particularly the hierarchical network and specific reward structure used by the winning MIT team [14], has led to a renewed interest in the analysis of effort exerted by agents in networks. The winning team’s strategy, utilized a recursive incentive mechanism. Our results show that, in this case for example, too much reward sharing encourages managers to ‘free-ride’, i.e. spend more time recruiting or managing and not enough time searching or working, though we do not study network formation games here.

The literature on strategic social network formation games and organizational design is vast

[11, 10]. We use the Price of Anarchy (PoA) [12] to measure the sub-optimality in outcome efforts, as a function of network structure and incentives, due to the self interested nature of agents. In the network contribution games literature, the PoA has been investigated in different contexts. Anshelevich and Hoefer [2] consider a model where an agent’s contribution locally benefit the nodes who share an edge with him, and give existence and PoA results for pairwise equilibrium for different contribution functions. The PoA in cooperative network formation is considered by Demaine et al. [6], while Roughgarden [19], Garg and Narahari [8] have considered the question in a selfish network routing context. Our setting is different from all of these since in our model the strategies are the efforts of the agents, which distinguishes it from the network formation and selfish routing literature, and we use multiple levels of information and reward sharing and study utilities that are asymmetric even for the neighboring nodes in the network, which distinguishes itself from the network contribution games.

2 A Hierarchical Network Model of Influencer and Influencee

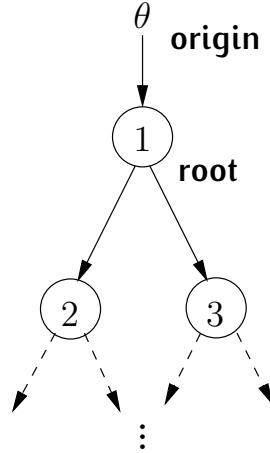


Figure 1: A typical hierarchical model.

In this section, we formalize a specific version of the hierarchical network model. Let $N = \{1, 2, \dots, n\}$ denote a set of agents who are connected over a hierarchy T . Each node i has a set of *influencers*, whose communication efforts influence his own direct payoff, and a set of *influencees*, whose direct payoffs are influenced by node i . In turn the production efforts of these influencees endow agent i with indirect payoffs. The origin (denoted by node θ) is a node assumed to be outside the network, and communicates perfectly with the first (root) node, denoted by 1.

We number nodes sequentially, so that each child has a higher index than his parent, thus the adjacency matrix is an upper triangular matrix with zeros on the diagonal. Figure 1 illustrates the model for an example hierarchical network.

The set of influencers of node i consists of the nodes (excluding node i) on the unique path from the origin to the node, and is denoted by $P_{\theta \rightarrow i}$. The set of influencees of node i consists of the nodes (again, excluding node i) in the subtree T_i below her.

The production effort, denoted by $x_i \in [0, 1]$, of node i yields a direct payoff to the node, and the particular way in which this occurs depends on its *productivity*. The remaining effort, $1 - x_i$, goes to communication effort, and improves the productivity of the influencees of the node. The constant sum of production effort and communication effort models the constraint on an agent’s time, and

therefore it is enough to write both the direct and indirect payoff of a node as a function of the production effort x_i . In particular, the productivity of a node, denoted by $p_i(\mathbf{x}_{P_{\theta \rightarrow i}})$, depends on the communication effort (and thus the production effort) of the influencers on path $P_{\theta \rightarrow i}$ to the node. The production effort profile of these influences is denoted by $\mathbf{x}_{P_{\theta \rightarrow i}}$.

It is useful to associate $x_i p_i(\mathbf{x}_{P_{\theta \rightarrow i}})$ with the value from the *direct output* of node i . The payoff to node i comprises two additive terms that capture:

- (1) the direct payoff, which depends on the value generated by the direct output of a node and the cost of production and communication effort, and is modulated by the productivity of the node, and
- (2) the indirect payoff, which is a fraction of the value associated with the direct output of any influencee j of the node.

Taken together, the payoff to a single node i is:

$$u_i(x_i, x_{-i}) = p_i(\mathbf{x}_{P_{\theta \rightarrow i}})f(x_i) + \sum_{j \in T_i \setminus \{i\}} h_{ij} p_j(\mathbf{x}_{P_{\theta \rightarrow j}}) x_j. \quad (1)$$

The first term is the product of the direct payoff and a function $f(x_i)$ (which models production output and cost) and captures the trade-off between direct output and cost of production and communication effort. The second term is the total indirect payoff received by node i due to the output $p_j(\mathbf{x}_{P_{\theta \rightarrow j}})x_j$ of its influencees. We insist that the productivity $p_j(\cdot)$ of any node j is non-decreasing in the communication effort of each influencer, and thus non-increasing in the production effort of each influencer, and we require $\frac{\partial}{\partial x_i} p_j(\mathbf{x}_{P_{\theta \rightarrow j}}) \leq 0$ for all nodes j , where i is an influencer of j .

Each node i receives a share h_{ij} of the value of the direct output of influencee j . The model can also capture a setting where an agent can only share output he creates, i.e. the total fraction of the output an agent retains and shares with the influencers is bounded at 1. Let us assume that agent j retains a share s_{jj} and shares s_{ij} with influencers $i \in P_{\theta \rightarrow j}$. A budget-balance constraint on the amount of direct value that can be shared requires $\sum_{i \in P_{\theta \rightarrow j} \cup \{j\}} s_{ij} \leq 1$. Assume that $s_{jj} = \gamma > 0$, for all j , so that each node retains the same fraction γ of its direct output value. Then, the earlier inequality can be written as, $\sum_{i \in P_{\theta \rightarrow j}} \frac{s_{ij}}{\gamma} \leq \frac{1}{\gamma} - 1$. By now defining $h_{ij} = \frac{s_{ij}}{\gamma}$, then the whole system is scaled by a factor γ . In addition to notational cleanliness, this transformation gives the advantage of not having any upper bound on the $\sum_{i \in P_{\theta \rightarrow j}} h_{ij}$, since any finite sum can always be accommodated with a proper choice of γ . Let us call the matrix $H = [h_{ij}]$ containing all the reward shares as the *reward sharing scheme*.

To highlight our results, we focus on a specific form of the payoff model, namely the *Exponential Productivity* (EP) model. A model is an instantiation of the direct-payoff function $f(x_i)$ and the productivity function $p_i(\cdot)$. In particular, in the EP model:

$$f(x_i) = x_i - \frac{x_i^2}{2} - b \frac{(1 - x_i)^2}{2}, \quad (2)$$

$$p_i(\mathbf{x}_{P_{\theta \rightarrow i}}) = \prod_{k \in P_{\theta \rightarrow i}} \mu(C_k) e^{-\beta x_k}, \quad (3)$$

where $b \geq 0$ is the cost of communication, C_k is the number of children of node k , function $\mu(C_k) \in [0, 1]$ required to be non-increasing, and $\beta \geq 0$ denotes the quality of communication, with higher β corresponding to a lower quality of communication. We assume $p_1 = 1$ for the root node. This models the root having perfect productivity. We interpret the term $\mu(C_k) e^{-\beta x_k}$ as the communication influence of node k on the agents in his subtree, and this takes values in $[0, 1]$.

The direct payoff of an agent i is quadratic in production effort x_i , and reflects a linear benefit x_i from direct production effort but a quadratic cost $x_i^2/2$ for effort. This is related to the bilinear payoff model of Ballester et al. [3]. However, here we also consider the cost due to communication, captured by $b(1 - x_i)^2/2$.

The productivity of node j , given by $p_j(\mathbf{x}_{P_{\theta \rightarrow j}})$, where $j \in T_i \setminus \{i\}$ warrants careful observation. Here we explain the components of this function and the reasons for choosing them. Consider $\mu(C_k)$, which is non-increasing in the number of children. C_k captures the idea that the effect of the communication effort is reduced if the node has more children to communicate with. An increase in production effort x_k reduces the productivity of influencees of node k . In particular, the exponential term in the productivity captures two effects: (a) a linear decrease in production effort gives exponential gain in the productivity of influencee, which captures the importance of communication and management in organizations [1]. Smaller values of β model better communication and a stronger positive effect on an influencee. (b) We can approximate other models by choosing β appropriately. Linear productivity corresponds to small values of β . This property is useful when the effects of production and communication on the payoff are equally important. For large β there is very small communication quality between agents and the value of communication effort is low.

The successive product of these exponential terms in the path from root to a node reflects the fact that a change in the production effort of an agent affects the productivity of the entire subtree below her. We note that the productivity of node j , where $j \in T_i \setminus \{i\}$, is not a concave function of x_i , leading to the payoff function u_i to be non-concave in x_i . Hence the existence of a Nash equilibrium is not guaranteed a priori through known results on concave games [18]. In the next section we will demonstrate the required conditions on existence and uniqueness of a Nash equilibrium. For brevity of notation, we will drop the arguments of productivity p_i at certain places where it is understood.

Our results on existence, uniqueness and their interpretations generalize to other network structures beyond hierarchies, which we show in the later part of the paper. However, despite the mathematical simplicity of the EP model, it allows for obtaining interesting results on the importance of influence, both communication and incentives, and gives insight on outcome efforts in a networked organization.

2.1 Main Results

The effect of communication efforts between nodes i and j , where $i \in P_{\theta \rightarrow j}$ is captured by the fractional productivity $\frac{p_i}{p_j}$ defined as, $p_{ij}(\mathbf{x}_{P_{i_- \rightarrow j}}) = \prod_{k \in P_{i_- \rightarrow j}} \mu(C_k) e^{-\beta x_k}$, (the node i_- is the parent of i in the hierarchy). This term is dependent only on the production efforts in the path segment between i and j and accounts for ‘local’ effects. We show in the following theorem that the Nash equilibrium production effort of node i depends on this local information from all its descendants.

Theorem 1 (Existence of a Nash Equilibrium) *A Nash equilibrium always exists in the effort game in the EP model, and is given by the production effort profile (x_i^*, x_{-i}^*) that satisfies,*

$$x_i^* = \left[1 - \frac{\beta}{1+b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij}(\mathbf{x}_{P_{i_- \rightarrow j}}^*) x_j^* \right]^+ \quad (4)$$

Proof: The proof of this theorem uses the Brouwer’s fixed point theorem and is given in Appendix A. \square

This theorem shows that the EP model allows us to guarantee the existence of (at least one) Nash equilibrium. In particular, we can make certain observations on the equilibrium production effort, some of which are intuitive.

- If communication improves, i.e., β becomes small, the production effort of each node increases.
- If the cost of management b increases, the production effort of each node increases.
- When reward sharing (h_{ij}) is large, agents reduce production effort and focus more on communication effort, i.e. nodes may over manage or ‘free-ride’.
- The computation of a Nash equilibrium at any node depends only on the production efforts of the nodes in its subtree. Thus, we can employ a backward induction algorithm which exploits this property that helps in an efficient computation of the equilibrium (this will be shown formally in the corollaries later in this section).

We turn now to establishing conditions for the uniqueness of this Nash equilibrium. Let us define the maximum amount of reward share that any node i can accumulate from a hierarchy T given a reward sharing scheme H as, $h_{\max}(T) = \sup_i \sum_{j \in T_i \setminus \{i\}} h_{ij}$. We also define the *effort update function* as follows.

Definition 1 (Effort Update Function (EUF)) *Let the function $F : [0, 1]^n \rightarrow [0, 1]^n$ be defined as,*

$$F_i(\mathbf{x}) = \left[1 - \frac{\beta}{1+b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij}(\mathbf{x}_{P_{i \rightarrow j}}) x_j \right]^+.$$

Note that the RHS of the above expression contains the production efforts of all the agents in the subtree of agent i . This function is a prescription of the choice of the production effort of agent i , if the agents below the hierarchy choose a certain effort profile. Hence the name ‘effort update’.

Theorem 2 (Sufficiency for Uniqueness) *If $\beta < \sqrt{\frac{1+b}{h_{\max}(T)}}$, the Nash equilibrium effort profile (x_i^*, x_{-i}^*) is unique and is given by Equation (4).*

Proof: The proof of this theorem shows that F is a contraction, and is given in Appendix A. \square

Theorem 3 (Tightness) *The sufficient condition of Theorem 2 is tight.*

Proof: Consider a 3 node hierarchy with nodes 2 and 3 being the children of node 1 (Figure 2). We show that if the sufficient condition is just violated, it results in multiple equilibria. Let $b = 0$, and $h_{12} = h_{13} = 0.25$, therefore $h_{\max}(T) = 0.25$. Theorem 2 requires that $\beta < 1/\sqrt{0.25} = 2$. We choose $\beta = 2$. The equilibrium efforts for node 2 and 3 are 1. Node 1 solves the following equation to find the equilibria.

$$1 - x_1 = e^{-2x_1}.$$

This equation has multiple solutions, $x_1 = 0, 0.797$, showing non-uniqueness. \square

The uniqueness condition indicates that the communication quality needs to be ‘good enough’ (small β) to ensure uniqueness of an equilibrium. It is worth noting that the uniqueness condition ensures the convergence of the best response dynamics, in which all the players start from any arbitrary effort profile \mathbf{x}_{init} , and sequentially update their efforts via the function F , to the unique equilibrium. This is a consequence of the fact that F is a contraction.

We now turn to the computational complexity of a Nash equilibrium. If there are multiple NE, these complexity results hold for computing a NE. Recall that the equilibrium computation of an agent requires only the production efforts and the reward structure of its subtree, and we can take advantage of the backward induction. This observation leads to the following corollaries.

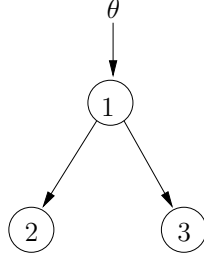


Figure 2: Tightness of the sufficiency (Theorem 2).

Corollary 1 *The worst-case complexity of computing the equilibrium effort for node i is $O(|T_i|^2)$. As a result, The worst-case complexity of computing the equilibrium efforts of the whole network is $O(n^2)$.*

Proof: To compute the equilibrium production effort x_i^* , node i needs to compute Equation (4). This requires to compute the equilibrium efforts for each node in his subtree T_i . Because of the fact that x_i^* depends only on the equilibrium efforts of the subtree below i , we can apply the backward induction method starting from the leaves towards the root of this sub-hierarchy T_i . The worst-case complexity of such a backward induction occurs when the sub-hierarchy is a line. In such a case the complexity would be $|T_i|(|T_i| - 1)/2 = O(|T_i|^2)$. In order to compute the equilibrium efforts of the whole network, it is enough to determine the equilibrium effort at the root because this would, in the process, determine the equilibrium efforts of each node in the hierarchy. This is also a consequence of the backward induction method of computing the equilibrium. The worst-case complexity of finding the equilibrium effort at the root is $O(n^2)$ and therefore the worst-case complexity of computing the equilibrium efforts of the whole network is also $O(n^2)$. \square

Given the characterization of the Nash equilibrium above, we now move on to questions of characterizing the amount of direct output value generated in equilibrium.

3 On the Price of Anarchy

In this section, we look at how the equilibrium effort level \mathbf{x}^* performs in comparison to the socially optimal outcome \mathbf{x}^{OPT} . We define the optimal effort level as the one that maximizes the *social output* of the hierarchy T :

$$SO(\mathbf{x}, T) = \sum_{i \in N} p_i(\mathbf{x}_{P_{\theta \rightarrow i}}) x_i \quad (5)$$

Therefore, $\mathbf{x}^{\text{OPT}} \in \arg \max_{\mathbf{x}} SO(\mathbf{x}, T)$. This is the direct production effort profile across the network that maximizes the total direct output value, considering also the effect of communication effort (induced by lower production effort) on the productivity of other nodes.

We will consider cases where the equilibrium is unique, hence, the *price of anarchy* is given by:

$$\text{PoA} = \frac{SO(\mathbf{x}^{\text{OPT}}, T)}{SO(\mathbf{x}^*, T)}. \quad (6)$$

The goal of this section is to understand the effect of reward sharing schemes on the PoA for different network structures and parameters. The following theorem shows that if the reward sharing is not properly designed, the PoA can be arbitrarily large. We first consider a single-level hierarchy (see Figure 3). To simplify the analysis, we also assume that the function $\mu(C_1) = 1$, irrespective of the number of children of node 1. By symmetry, we consider a single value h , such that $h_{12} =$

$h_{13} = \dots = h_{1n} = h$. We refer to this model as **FLAT**. We will return to this model later as well, after presenting our results for more general balanced hierarchies. We first consider what happens when there is bad communication (β large) and no profit sharing ($h = 0$), between node 1 and its children.

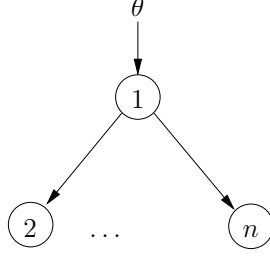


Figure 3: FLAT hierarchy.

Theorem 4 (Large PoA) For $n \geq 3$, the PoA is $\frac{n-1}{2}$ in the FLAT hierarchy when $\beta = \ln(n-1)$ and $h = 0$.

Proof: For FLAT, the social output is given by, $SO(\mathbf{x}, \text{FLAT}) = \sum_{i=2}^n e^{-\beta x_i} x_i + x_1$. We see that $\beta = \ln(n-1) \geq -\ln(1 - \frac{1}{n-1})$, for all $n \geq 3$. The optimal effort profile $\mathbf{x}^{\text{OPT}} = (0, 1, \dots, 1)$ maximizes the social output (stated in Corollary 2, for the proof see Lemma 6 in Appendix B). Hence the optimal social output is $n-1$. However, for reward sharing factor $h = 0$, we get the equilibrium effort profile from Equation (4) to be $\mathbf{x}^* = (1, 1, \dots, 1)$. This yields a social output of $(n-1)e^{-\beta} + 1$. Hence the PoA is $\frac{n-1}{(n-1)e^{-\ln(n-1)} + 1} = \frac{n-1}{2}$. \square

However, if h is chosen appropriately, e.g. if it were chosen to be large positive, the equilibrium effort profile given by Equation (4) would have been closer to that of the optimal. Hence PoA could have been reduced and made closer to 1.

This raises a natural question: *is it always possible to design a suitable reward sharing scheme that can make PoA = 1 for any given hierarchy?* In order to answer that, we define the *stability* of an effort profile \mathbf{x} .

Definition 2 (Stable Effort Vector) An effort profile $\mathbf{x} = (x_1, \dots, x_n)$ is stable, represented by $\mathbf{x} \in S$, if $\mathbf{x} \geq \mathbf{0}$, and there exists a reward sharing matrix $H = [h_{ij}]$, $h_{ij} \geq 0$, such that,

$$\sum_{j \in T_i \setminus \{i\}} a_{ij}(\mathbf{x}) h_{ij} \geq 1 - x_i; \quad \sum_{j \in T_i \setminus \{i\}} h_{ij} \leq \frac{1+b}{\beta^2}, \quad \forall i \in N. \quad (7)$$

Where, $a_{ij}(\mathbf{x}) = \frac{\beta}{1+b} p_{ij}(\mathbf{x}_{P_{i \rightarrow j}}) x_j$, for all $j \in T_i \setminus \{i\}$, and zero otherwise.

The inequalities capture a required balance between incentives and information flow. In the first inequality, for a fixed communication factor β and cost coefficient b , the term $a_{ij}(\cdot)$ is proportional to the fractional output (fractional productivity \times production effort) of an agent j . After multiplying with h_{ij} , this is the effective indirect output that i receives from j . The RHS of the inequality can be interpreted as the communication effort of agent i . Hence, this inequality says that the total indirect benefit should be at least equal to the effort put in by a node for communicating the information to its subtree. If we consider that the agents share information based on the reward share they receive, the flow of information and reward forms a closed loop. The second inequality says that the closed loop ‘gain’ of the information flow (β^2) and the reward share accumulated by agent i ($\sum_{j \in T_i \setminus \{i\}} h_{ij}$) should be bounded by the cost of sharing the information. The closed loop ‘gain’ is essentially the reward that an agent accumulates due to his communication effort *through* his descendants. We can connect a stable effort vector with the Nash equilibrium of the effort game.

Lemma 1 (Stability-Nash Relationship) *If an effort profile $\mathbf{x} = (x_1, \dots, x_n)$ is stable, it is the unique Nash equilibrium of the effort game.*

Proof: Let \mathbf{x} is a stable effort profile. So, there exists a reward sharing matrix $H = [h_{ij}]$, $h_{ij} \geq 0$, s.t. Equation (7) is satisfied with \mathbf{x} . Also $\mathbf{x} \geq \mathbf{0}$. Therefore, reorganizing the first inequality of Equation (7) and noting the fact that $x_i \geq 0$, $\forall i \in N$, we get,

$$x_i = \left[1 - \sum_{j \in T_i \setminus \{i\}} a_{ij}(\mathbf{x}) h_{ij} \right]^+, \quad \forall i \in N.$$

Under the condition given by the second inequality of Equation (7), the Nash equilibrium is unique and is given by the above expression (recall Theorem 2). Hence, \mathbf{x} is the unique Nash equilibrium of this game. \square

Now it is straightforward to see that the stability of \mathbf{x}^{OPT} is sufficient for PoA to be 1. Hence, we can write the following lemma.

Lemma 2 (No Anarchy) *Any reward sharing matrix that makes \mathbf{x}^{OPT} stable provides a PoA of 1.*

An important question is then: *how efficiently can we check if a given effort profile \mathbf{x} is stable or not?* The answer is that we can solve the following feasibility linear program (LP) for a given effort profile:

$$\begin{aligned} \min \quad & 1 \\ \text{s.t.} \quad & \left. \begin{aligned} \sum_{j \in T_i \setminus \{i\}} a_{ij}(\mathbf{x}) h_{ij} &\geq 1 - x_i, \\ \sum_{j \in T_i \setminus \{i\}} h_{ij} &\leq \frac{1+b}{\beta^2}, \\ h_{ij} &\geq 0, \forall j, \end{aligned} \right\} \quad \forall i \in N. \end{aligned} \quad (8)$$

If a solution exists to the above LP, we conclude that \mathbf{x} is stable. Linear programs can be efficiently solved and therefore checking an effort profile for stability can be done efficiently.

3.1 Effect of communication on the PoA in general hierarchies

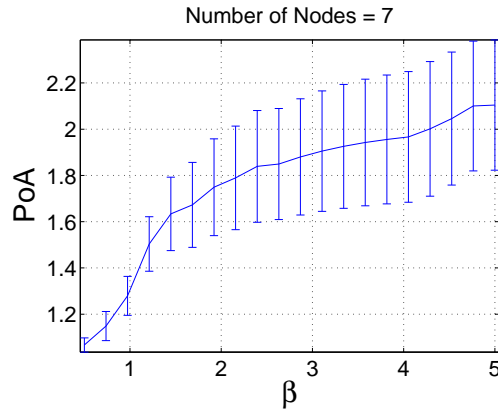


Figure 4: PoA as a function of communication factor β .

In the previous section, we have seen that for a given communication factor β , one can determine if there exists a reward sharing scheme H for a hierarchy that makes the $\text{PoA} = 1$. We are also interested in understanding how the PoA depends upon the communication factor β , when such an

H does not exist, that is, if feasibility LP (Equation (8)) does not return a feasible H and $\mathbf{x}^{\text{OPT}} \notin S$. In such a scenario, we cannot guarantee PoA to be unity.

For any given reward sharing matrix H , there is an equilibrium effort profile $\mathbf{x}^*(H)$. We can thus solve for $H_{\max} \in \arg \max_{H: \mathbf{x}^*(H) \in S} SO(\mathbf{x}^*(H))$, which leads to an equilibrium effort profile $\mathbf{x}^*(H_{\max})$ that is stable and maximizes the social output. When we cannot find a reward sharing scheme to set $\text{PoA} = 1$, H_{\max} is the choice of reward sharing that minimizes the PoA.

Figure 4 shows a simulation where for each β we generated a large number of random 7 node hierarchical networks. For each choice, we found the optimal reward sharing matrix H_{\max} . The plot shows the mean PoA with standard error around it. We see from the simulation that as β increases, it limits the reward share among the agents (second inequality of Equation (7)). This shrinks the set of stable effort profiles S , and gives rise to an increase in the PoA. This again highlights the importance of efficient communication in organizational hierarchies. In the next section we make this intuition more formal, by deriving bounds on the PoA for such general (possibly) unstable hierarchies.

3.2 Price of Anarchy in Balanced Hierarchies

In this section we consider a simple yet representative class of hierarchies, namely the balanced hierarchies, and analyze the effect of communication on PoA and provide efficient bounds. Hierarchies in organizations are often (nearly) balanced, and the FLAT or linear networks are special cases of the balanced hierarchy (depth = 1 or degree = 1). Hence, the class of balanced hierarchies can generate useful insights. In addition, the symmetry in balanced hierarchies allows us to obtain interpretable closed-form bounds and understand the relative importance of different parameters.

We consider a balanced d -ary tree of depth D . By symmetry, the efforts of the nodes that are at the same level of the hierarchy are same at both equilibrium and optimality. This happens because of the fact that in the EP model, both the equilibrium and optimal effort profile computation follows a backward induction method starting from the leaves towards the root. Since the nodes in the same level of the hierarchy is symmetric in the backward induction steps, they have identical effort profiles.

With a little abuse of notation, we denote the efforts of each node at level i by x_i . We start numbering the levels from root, hence, there are $D + 1$ levels. Note that there are a few interesting special cases of this model, namely (a) $d = 2$: balanced binary tree, (b) $D = 1$: flat hierarchy, (c) $d = 1$: line. We assume, for notational simplicity only, that the function $\mu(C_k) = 1$, for all C_k , though our results generalize. This function is the coefficient of the productivity function. $\mu(C_k) = 1$ also models organizations where each manager is assigned a small team and there is no attenuation in productivity due to the number of children. In order to present the *price of anarchy* (PoA) results, we define the set ξ :

$$\xi(\beta) = \left\{ x : x = \left[1 - \frac{1}{\beta} e^{-\beta x} \right]^+ \right\}. \quad (9)$$

This set is the set of possible equilibrium effort levels for agents at the penultimate level of the EP model hierarchy when $\beta > 1$. Note that this set is a singleton, when $\beta > 1$. Depending on β , we define a lower bound $\phi(d, \beta)$ on the contribution of an agent toward the social output, and a sequence of nested functions t_i , where d is the degree of each node.

$$\begin{aligned} \phi(d, \beta) &= \max \left\{ \frac{1}{\beta} (1 + \ln(d\beta)), d\beta + (1 - d\beta)\xi(\beta) \right\}, \\ t_1(d, \beta) &= \phi(d, \beta), t_2(d, \beta) = \phi(d \cdot \phi(d, \beta), \beta), \dots, t_D(d, \beta) = \phi(d \cdot t_{D-1}(d, \beta), \beta). \end{aligned} \quad (10)$$

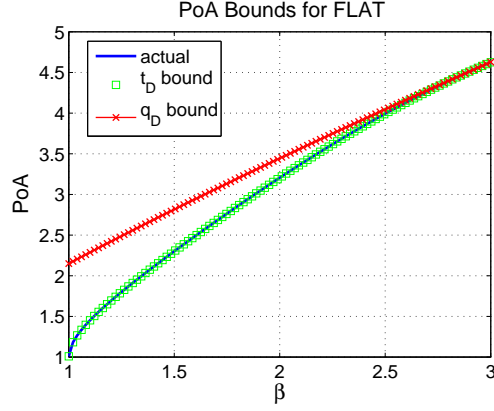


Figure 5: Bounds on PoA for FLAT, $d = 6, D = 1$.

Theorem 5 (Price of Anarchy) *For a balanced d -ary hierarchy with depth D , as β increases, we can show the following price of anarchy results.*

$$\begin{aligned} \text{When } 0 \leq \beta \leq 1, \quad & PoA = 1, \\ \text{and when } 1 < \beta < \infty, \quad & PoA \leq \frac{d^D}{t_D(d, \beta)}. \end{aligned} \tag{11}$$

Proof: See Appendix B. □

As opposed to our choice of lower bound ϕ , a naïve lower bound of $\frac{1}{\beta}(1 + \ln(d\beta))$ can also be used. However, this gives a weaker bound for any hierarchy. As an example, we demonstrate the weakness for FLAT (recall Figure 3) in Figure 5 (the FLAT hierarchy is a balanced tree with $D = 1, d = n - 1$). Figure 5 shows that the bound given by our analysis is tight for FLAT, indicating the value of the analysis and also gives intuition to the shape of the effect of β on the PoA.

We can then have the following corollaries of Theorem 5,

Corollary 2 (Optimal Effort) *For the FLAT hierarchy, if $0 \leq \beta < -\ln(1 - \frac{1}{n})$, the optimal effort profile is where all nodes put unit effort. When $-\ln(1 - \frac{1}{n}) \leq \beta < \infty$, the optimal changes to the profile where the root node puts zero effort and each other node puts unit effort.*

Corollary 3 *For the FLAT hierarchy, when $0 \leq \beta \leq 1$, $PoA = 1$, and when $1 < \beta < \infty$, $PoA \leq \frac{n}{\phi(d, \beta)}$.*

The second corollary above makes rigorous the intuition that when β is small enough the optimal \mathbf{x} can be achieved by choosing a small enough reward share h . However, when β grows, in order to ensure uniqueness of the Nash equilibrium, the choice of h becomes limited (as it has to satisfy $\leq (1 + b)/\beta^2$) resulting in a PoA, as captured in Figure 5.

4 A General Network Model of Influencer and Influencee

In this section, we show that the results on existence and uniqueness of a pure strategy Nash equilibrium generalize to a much broader setting of agents as influencer and influencees interacting over an arbitrary network.

Suppose that the agents are connected over a (possibly non-hierarchical) network G . Each node i has a set of influencers, denoted by R_i (generalizing $P_{\theta \rightarrow i}$), and a set of influencees, E_i (generalizing $T_i \setminus \{i\}$). We import the notation from Section 2 with their exact or analogous meanings for productivity $p_i(\mathbf{x}_{R_i})$ and reward sharing scheme H . Now, the payoff function of agent i is given by,

$$u_i(x_i, x_{-i}) = p_i(\mathbf{x}_{R_i})f(x_i) + \sum_{j \in E_i} h_{ij}p_j(\mathbf{x}_{R_j})x_j. \quad (12)$$

We assume that f is a strictly concave function, and is continuously differentiable. We will refer to the product of effort x_i and productivity $p_i(\mathbf{x}_{R_i})$ as the output, and denote it by y_i . In this context, we do not impose any condition on the nature of the productivity function $p_i(\cdot)$, and as before, this game is also not necessarily a concave game and the existence of a Nash equilibrium is not always guaranteed.

4.1 Results

The payoff function given by Equation (12) induces a game between the influencers and the influencees. In addition, as before, every agent faces a trade-off when deciding how much production and communication effort to exert. We will need the following facts.

Fact 1 *If a function is continuously differentiable and strictly concave, its derivative is continuous and monotone decreasing.*

Fact 2 *A continuous and monotone decreasing function is invertible and the inverse is also continuous and monotone decreasing.*

Using the above two facts, we see that the inverse of f' exists and is monotone decreasing. Let us denote f'^{-1} by ℓ . Let us define two functions g and T similar to that defined in Section 2.

$$g_i(\mathbf{x}) = \sum_{j \in E_i} h_{ij} \left(-\frac{1}{p_i(\mathbf{x}_{R_i})} \frac{\partial p_j(\mathbf{x}_{R_j})}{\partial x_i} x_j \right) \Big|_{\mathbf{x}}, \quad (13)$$

$$T(x) = \min\{\max\{0, x\}, 1\}. \quad (14)$$

Fact 3 *The function T is continuous.*

Lemma 3 (Necessary condition for Nash equilibrium) *If a Nash equilibrium exists for the effort game in a influencer-influencee network, the effort profile (x_i^*, x_{-i}^*) must satisfy,*

$$x_i^* = T \circ \ell \circ g_i(\mathbf{x}^*), \quad \forall i \in N. \quad (15)$$

To illustrate what this necessary condition means, let us assume, for simplicity, that we do not hit the edges of the truncation function T . Therefore we can rewrite Equation (15) as,

$$f'(x_i^*) = \sum_{j \in E_i} h_{ij} \left(-\frac{1}{p_i} \frac{\partial p_j}{\partial x_i} x_j \right) \Big|_{\mathbf{x}^*} = \sum_{j \in E_i} h_{ij} \left(-\frac{1}{p_i} \frac{\partial y_j}{\partial x_i} \right) \Big|_{\mathbf{x}^*} \quad (16)$$

Where $y_j = p_j x_j$ is the output of node j . We have dropped the arguments of p_i and p_j for brevity of notation. The expression on the LHS is the rate of change of direct benefit for agent i . The RHS is the rate at which the passive output of agent i changes w.r.t. his effort x_i and productivity p_i . If the LHS is larger, the agent would gain more at the margin by increasing x_i . This is because the derivative $(-\partial y_j / \partial x_i)$ is non-negative since $\partial p_j / \partial x_i$ is always non-positive. Similarly, if the RHS was larger, the agent could gain at the margin by decreasing x_i . Hence Equation (16) resembles a rate balance equation (or demand-supply curve) where the rate of effective direct payoff matches the rate of passive payoffs.

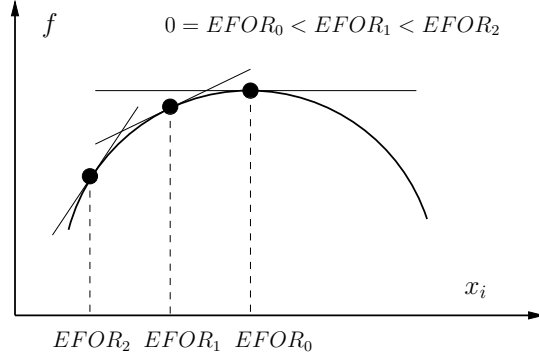


Figure 6: Free riding phenomenon of a node having higher EFOR from the influencees.

Let us define the *effective fractional output rate* (EFOR) at \mathbf{x} as $\sum_{j \in E_i} h_{ij} \left(-\frac{1}{p_i} \frac{\partial y_j}{\partial x_i} \right) \Big|_{\mathbf{x}}$. In some settings, e.g., if $p_i(\mathbf{x}_{R_i}) = \prod_{k \in R_i} (1 - x_k)$, the fractional output rate $\left(-\frac{1}{p_i} \frac{\partial y_j}{\partial x_i} \right)$ can be independent of the production effort of i , i.e., x_i . In such settings, Equation (16) shows that if the EFOR of node i increases, the equilibrium for node i will move in a direction that decreases production effort x_i . This happens because the slope of f in equilibrium is always non-negative and its increase leads to a smaller x_i because of the concavity of f . This phenomenon is graphically shown in Figure 6. This shows that the nodes having a higher EFOR, which is a function of the network position of an agent, can *free-ride* on the production efforts of the influencees.

Following Definition 1, we define the *effort update function* (EUF) $F : [0, 1]^n \rightarrow [0, 1]^n$ for the general setting as, $F(\mathbf{x}) = T \circ \ell \circ g(\mathbf{x})$, where $T \circ \ell$ operates on the vector function g element-wise. Therefore, the question of existence of a Nash equilibrium of this effort game is the same as asking the question if the following fixed point equation has a solution: $\mathbf{x} = F(\mathbf{x})$.

In the following, we provide a sufficient condition for existence of the Nash equilibrium, and its uniqueness.

Lemma 4 For $p_i > 0$, for all $i \in N$, and continuously differentiable, $F(\mathbf{x})$ is continuous.

Proof: Given $p_i > 0$, for all $i \in N$, and is continuously differentiable. Therefore, the function g , defined in Equation (13), is continuous in \mathbf{x} . Using Facts 2 and 3, we see that the functions ℓ and T are continuous. Hence, $F \equiv T \circ \ell \circ g$ is continuous in \mathbf{x} . \square

Theorem 6 (Sufficient Condition for a Nash Equilibrium) For $p_i > 0$, for all $i \in N$, and continuously differentiable, the effort game has at least one Nash equilibrium.

Proof: The Nash equilibrium of the effort game is same as the fixed point of the equation, $\mathbf{x} = F(\mathbf{x})$. Since F is continuous (Lemma 4), Brouwer's fixed point theorem immediately ensures a fixed point of the above equation to exist. Hence, the effort game has at least one Nash equilibrium. \square

Let us use the shorthand $G \equiv \ell \circ g$. The following theorem provides a sufficient condition for the uniqueness of the Nash equilibrium.

Theorem 7 (Sufficient Condition for unique Nash Equilibrium) If $\sup_{\mathbf{x}_0} |\nabla G(\mathbf{x}_0)| < 1$, then the Nash equilibrium effort profile (x_i^*, x_{-i}^*) is unique and is given by Equation (15).

Proof: The key here is to show that F is a contraction. We follow the steps of Theorem 2 as follows:

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \leq \|G(\mathbf{x}) - G(\mathbf{y})\| \leq |\nabla G(\mathbf{x}_0)| \cdot \|\mathbf{x} - \mathbf{y}\|.$$

This is a contraction as $\sup_{\mathbf{x}_0} |\nabla G(\mathbf{x}_0)| < 1$. \square

4.1.1 Interpretation of the sufficient condition of the uniqueness

The sufficient condition given by Theorem 7 is a technical one. We now discuss an interesting geometric interpretation of this condition. By the Taylor expansion of G with first order remainder term, we get,

$$G(\mathbf{x}) - G(\mathbf{y}) = \nabla G(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{y}).$$

Where \mathbf{x}_0 lies on the line joining \mathbf{x} and \mathbf{y} . Using singular value decomposition, we get, $\nabla G(\mathbf{x}_0) = U_0 \Sigma_0 V_0^\top$. Therefore, for each pair of points \mathbf{x} and \mathbf{y} , we can transform the space of efforts with a pure rotation as follows.

$$\begin{aligned} G(\mathbf{x}) - G(\mathbf{y}) &= U_0 \Sigma_0 V_0^\top \cdot (\mathbf{x} - \mathbf{y}), \\ \Rightarrow U_0^\top (G(\mathbf{x}) - G(\mathbf{y})) &= \Sigma_0 \cdot (\bar{\mathbf{x}} - \bar{\mathbf{y}}), \text{ where } \bar{\mathbf{x}} = V_0^\top \mathbf{x}, \bar{\mathbf{y}} = V_0^\top \mathbf{y} \\ \Rightarrow R(\bar{\mathbf{x}}) - R(\bar{\mathbf{y}}) &= \Sigma_0 \cdot (\bar{\mathbf{x}} - \bar{\mathbf{y}}), \text{ where } R \equiv U_0^\top G V_0. \end{aligned}$$

Hence, for any pair of points \mathbf{x} and \mathbf{y} , we can rotate the space so that the effect of the deviation to \mathbf{x} from \mathbf{y} can be captured by a weight on each of the coordinates in the rotated space. Here, the diagonal matrix Σ_0 contains the weights along its diagonal.

Theorem 7 says that for any point \mathbf{x}_0 , if the absolute value of all the elements of this diagonal matrix is smaller than unity, the uniqueness of the Nash equilibrium is guaranteed. Let us denote the rotated vector of \mathbf{x}_0 by $\mathbf{z}_0 := V_0^\top \mathbf{x}_0$. The diagonal elements can be written w.r.t. the vectors in the rotated space as,

$$(\Sigma_0)_{ii} = \sum_{j \in E_i} h_{ij} \left(-\frac{1}{p_i} \frac{\partial^2 p_j}{\partial z_i^2} z_j \right) \Big|_{\mathbf{z}_0} = \frac{\partial \text{EFOR}(\mathbf{z}_0)}{\partial z_{i,0}}.$$

In other words, the diagonal elements are the rate of change of EFOR at \mathbf{z}_0 . Having the rate of change of EFOR bounded by 1 is a sufficient condition for a unique Nash equilibrium. One can think of the EFOR as the product of two effects: (1) the rate of change in productivity, which increases the payoff of the influencees, (2) the reward share h_{ij} 's. The sufficient condition essentially says that the net effect should not be too large in order to guarantee unique equilibrium.

Figure 7 shows a graphical illustration of the phenomenon in polar co-ordinates, where the directions represent that of the vectors. The results say that if for any vector, the singular values of the Jacobian matrix of G at that point lies entirely within the unit ball, then there exists a unique Nash equilibrium. This is similar to the feedback loop gain of a feedback controller, where the closed loop gain being smaller than unity ensures stability. We find this natural parallel between notions of stability (from control theory) and uniqueness of Nash equilibria interesting.

5 Conclusions and Future Work

In this paper, we build on the papers by Bramoullé and Kranton [5], Ballester et al. [3] and develop an understanding of effort levels in influence networks. Taking a game theoretic perspective, we consider a general utility model which results in a non-concave game, but are able to show results on the existence and uniqueness of Nash equilibrium efforts and when this Nash equilibrium can be reached by agents' best response dynamics. For hierarchical networks, we focused on the EP model where we develop closed form expressions and bounds on the PoA. These results give us the insight that communication in hierarchies should be good enough to allow for the design of efficient networks. At the same time, for a given network structure and communication level, the profit sharing has to be designed appropriately to minimize the PoA. If reward sharing is too high, then managers free-ride resulting in a large PoA, an observation we believe is novel to our work.

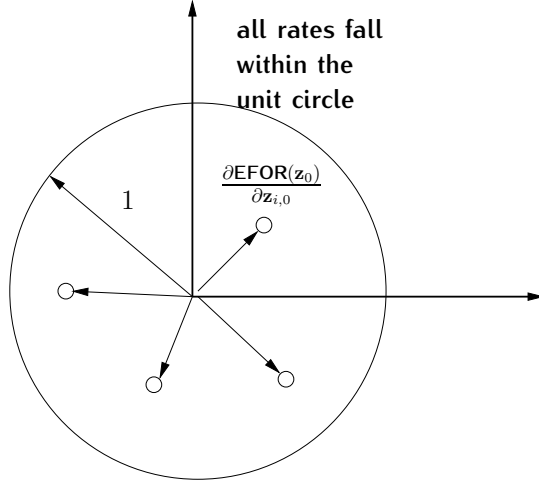


Figure 7: Geometric interpretation of the condition for uniqueness of the Nash equilibrium.

The connection between matrix stability and uniqueness of Nash equilibria that arose in our work, is of particular interest to us for future research. In particular, for the case of quadratic cost functions there was a direct interpretation of the uniqueness condition in terms of a Jacobian matrix stability. This stability property is directly related to the contraction property that shows that agents following local updates on effort levels will converge to the Nash equilibrium another desirable property. Pursuing these connections in the investigation of organizational network formation games is an important direction of future research.

APPENDIX

A Proofs for the Exponential Productivity Model

Proof of Theorem 1 Proof: First we show that if a Nash equilibrium profile (x_i^*, x_{-i}^*) exists for the effort game, it must satisfy Equation (4). For notational convenience, we drop the arguments of p_i and p_{ij} , which are functions of $\mathbf{x}_{P_{\theta \rightarrow i}}$ and $\mathbf{x}_{P_{i \rightarrow j}}$ respectively. Each agent $i \in N$ solves the following optimization problem.

$$\begin{aligned} \max_{x_i} \quad & u_i(x_i, x_{-i}) \\ \text{s.t.} \quad & x_i \geq 0 \end{aligned} \tag{17}$$

Combining Equations (1), (2), and (3), we get,

$$u_i(x_i, x_{-i}) = p_i(\mathbf{x}_{P_{\theta \rightarrow i}}) \left(x_i - \frac{x_i^2}{2} - b \frac{(1 - x_i)^2}{2} \right) + \sum_{j \in T_i \setminus \{i\}} h_{ij} p_j(\mathbf{x}_{P_{\theta \rightarrow j}}) x_j.$$

Note that we have relaxed the constraint from $0 \leq x_i \leq 1$. The first additive term in the utility function has the peak at $x_i = 1$. The second term has $e^{\beta x_i}$ in the p_j , which is decreasing in x_i . Therefore, the optimal x_i that maximizes this utility will be ≤ 1 . Hence, in this problem setting, the optimal solution for both the exact and the relaxed problems is the same. So, it is enough to consider the above problem. For this non-linear optimization problem, we can write down the Lagrangian as follows.

$$\mathcal{L} = u_i(x_i, x_{-i}) + \lambda_i x_i, \quad \lambda_i \geq 0.$$

The KKT conditions for this optimization problem (17) are:

$$\frac{\partial \mathcal{L}}{\partial x_i} = 0, \Rightarrow \frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) + \lambda_i = 0, \quad (18)$$

$$\lambda_i x_i = 0, \quad \text{complementary slackness.} \quad (19)$$

Case 1: $\lambda_i = 0$, then from Equation (18) we get,

$$\begin{aligned} & p_i(1 - x_i + b(1 - x_i)) + \sum_{j \in T_i \setminus \{i\}} h_{ij} \frac{\partial p_j}{\partial x_i} x_j = 0 \\ \Rightarrow & p_i(1 + b)(1 - x_i) - \beta \sum_{j \in T_i \setminus \{i\}} h_{ij} p_j x_j = 0 \\ \Rightarrow & 1 - x_i = \frac{\beta}{1 + b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j, \text{ with } p_{ij} \text{ as defined} \\ \Rightarrow & x_i = 1 - \frac{\beta}{1 + b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j. \end{aligned} \quad (20)$$

Case 2: $\lambda_i > 0$, then from Equation (19) we get $x_i = 0$, and from Equation (18), $\frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) < 0$. Carrying out the differentiation as in Equation (20) and combining with (21) we get,

$$0 = x_i > 1 - \frac{\beta}{1 + b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j. \quad (21)$$

$$x_i = \left[1 - \frac{\beta}{1 + b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j \right]^+.$$

Since this condition has to hold for all nodes $i \in N$, the equilibrium profile (x_i^*, x_{-i}^*) must satisfy the above equality. The question now is whether a Nash equilibrium exists at all for this effort game. Let us recall the function F from Definition 1, which is the composition of the following two functions.

$$g_i(\mathbf{x}) = 1 - \frac{\beta}{1 + b} \sum_{j \in T_i \setminus \{i\}} h_{ij} p_{ij} x_j, T(y) = \max(0, y).$$

We recall that, for the EP model, $p_{ij} = \prod_{k \in P_{i \rightarrow j}} \mu(C_k) e^{-\beta x_k}$. Therefore, the function g_i is continuous. Also, T is continuous, and $F_i \equiv T \circ g_i$. Therefore, F_i is also continuous, for all $i \in N$. We also know that if for all $i \in N$, F_i is continuous, then F is continuous too. Therefore, we can apply Brouwer's fixed point theorem to conclude that the fixed point equation $\mathbf{x} = F(\mathbf{x})$ has at least one solution. For the effort game, any fixed point of this equation is a Nash equilibrium. Hence the equilibrium always exists for the effort game with EP model, and is given by Equation (4). \square

Proof of Theorem 2

We prove this theorem via the following Lemma.

Lemma 5 *If $\beta < \sqrt{\frac{1+b}{h_{\max}(T)}}$, the function F is a contraction.*

Proof: The Taylor series expansion of g with a first order remainder term is as follows. There exists a point \mathbf{x}_0 that lies on the line joining \mathbf{x} and \mathbf{y} , such that,

$$g(\mathbf{x}) = g(\mathbf{y}) + \nabla g(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{y}).$$

Where, $\nabla g(\mathbf{x}_0)$ is the Jacobian matrix.

$$\nabla g(\mathbf{x}_0) = \left(\begin{array}{ccc} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{array} \right) \bigg|_{\mathbf{x}_0}$$

In order to show that F is a contraction, we note that F is a truncation of g . Hence, $\|F(\mathbf{x}) - F(\mathbf{y})\| \leq \|g(\mathbf{x}) - g(\mathbf{y})\|$, for all $\mathbf{x}, \mathbf{y} \in [0, 1]^n$. Let us consider the following term,

$$\|F(\mathbf{x}) - F(\mathbf{y})\| \leq \|g(\mathbf{x}) - g(\mathbf{y})\| \leq |\nabla g(\mathbf{x}_0)| \cdot \|\mathbf{x} - \mathbf{y}\| \quad (22)$$

Where the matrix norm $|\nabla g(\mathbf{x}_0)|$ is the largest singular value of the Jacobian matrix $\nabla g(\mathbf{x}_0)$. We see that in our special structure in the problem, this matrix is upper triangular, hence the diagonal elements are the singular values. Suppose, the k -th diagonal element yields the largest singular value.

$$\begin{aligned} |\nabla g(\mathbf{x}_0)| &= \left| \frac{\partial g_k}{\partial x_k} \right|_{\mathbf{x}_0} = \frac{\beta^2}{1+b} \sum_{j \in T_k \setminus \{k\}} h_{kj} p_{kj} x_j \bigg|_{\mathbf{x}_0} \\ \Rightarrow \sup_{x_0} |\nabla g(\mathbf{x}_0)| &\leq \frac{\beta^2}{1+b} \cdot h_{\max}(T) < 1, \quad \text{since } \beta^2 < \frac{1+b}{h_{\max}(T)}. \end{aligned}$$

The first inequality above holds due to the fact that p_{kj} 's and x_j 's are ≤ 1 , and by the definition of $h_{\max}(T)$. Hence, from Equation (22), we get that F is a contraction. \square

Proof: [of Theorem 2] Brouwer's fixed point theorem only guarantees the existence of a fixed point, which implies that the Nash equilibrium exists. Under the sufficient condition given by Lemma 5, the fixed point of $\mathbf{x} = F(\mathbf{x})$ is unique. Therefore, the Nash equilibrium is also unique, and is given by Equation (4). \square

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APPENDIX

B Proofs of the price of anarchy results in balanced hierarchies

Proof of Theorem 5

We prove this theorem via the following lemma, which finds out the optimal effort profile for β above a threshold.

Lemma 6 (Optimal Efforts) *For a balanced d -ary hierarchy with depth D , any optimal effort profile has $x_{D+1}^{OPT} = 1$. When $-\ln(1 - \frac{1}{d}) \leq \beta < \infty$, the optimal effort profile is $x_i^{OPT} = 0$, $\forall i = 1, \dots, D$, and $x_{D+1}^{OPT} = 1$.*

Proof: The social outcome for a given effort vector \mathbf{x} on the balanced hierarchy is as follows. Since, the hierarchy is understood here, we use $SO(\mathbf{x})$ instead of $SO(\mathbf{x}, \text{BALANCED})$.

$$SO(\mathbf{x}) = x_1 + de^{-\beta x_1}x_2 + d^2e^{-\beta(x_1+x_2)}x_3 + \dots + d^De^{-\beta(\sum_{i=1}^D x_i)}x_{D+1}.$$

It is clear that for any effort profile of the other nodes the effort at the leaves that maximizes the above expression is $x_{D+1} = 1$. This proves the first part of the lemma. Hence we can simplify the above expression by,

$$\begin{aligned} SO(\mathbf{x}) &= x_1 + de^{-\beta x_1}x_2 + d^2e^{-\beta(x_1+x_2)}x_3 + \dots + d^De^{-\beta(\sum_{i=1}^D x_i)} \\ &= x_1 + de^{-\beta x_1}x_2 + \dots + d^{D-1}e^{-\beta(\sum_{i=1}^{D-1} x_i)}(x_D + de^{-\beta x_D}) \\ &\leq x_1 + de^{-\beta x_1}x_2 + \dots + d^{D-1}e^{-\beta(\sum_{i=1}^{D-1} x_i)} \cdot d. \end{aligned} \tag{23}$$

The last inequality occurs since $\beta \geq -\ln(1 - \frac{1}{d})$, and $x_D = 0$ meets this inequality with a equality. Also since $\beta \geq -\ln(1 - \frac{1}{d})$ implies that $\beta \geq -\ln(1 - \frac{1}{d^k})$, for all $k \geq 2$, the next inequality will also be met by $x_{D-1} = 0$ as shown below.

$$\begin{aligned} SO(\mathbf{x}) &= x_1 + de^{-\beta x_1}x_2 + \dots + d^{D-1}e^{-\beta(\sum_{i=1}^{D-1} x_i)} \cdot d \\ &= x_1 + de^{-\beta x_1}x_2 + \dots + d^{D-2}e^{-\beta(\sum_{i=1}^{D-2} x_i)}(x_{D-1} + d^2e^{-\beta x_{D-1}}) \\ &\leq x_1 + de^{-\beta x_1}x_2 + \dots + d^{D-2}e^{-\beta(\sum_{i=1}^{D-2} x_i)} \cdot d^2. \end{aligned}$$

This inequality is also achieved by $x_{D-1} = 0$. We can keep on reducing the terms from the right in the RHS of the above equation, and in all the reduced forms, $x_i = 0, i = D-1, D-2, \dots, 1$ will maximize the social output expression. Hence proved. \square

Proof: [of Theorem 5] *Case 1* ($0 \leq \beta \leq 1$): From Lemma 6, $x_{D+1} = 1$ for optimal effort. However, for any equilibrium effort profile $x_{D+1} = 1$ as well. Therefore we consider the equilibrium effort of the nodes at level D .

$$x_D = 1 - \frac{\beta}{1+b} de^{-\beta x_D} h_{D,D+1}. \quad (24)$$

The constraint for unique equilibrium demands that $dh_{D,D+1} \leq (1+b)/\beta^2$, which makes $\frac{\beta}{1+b} dh_{D,D+1} \leq 1/\beta$, while $1/\beta \geq 1$. So, we have the liberty of choosing the right $h_{D,D+1}$ to achieve any $x_D \in [0, 1]$, and in particular, the x_D^{OPT} . We apply backward induction on the next level above.

$$x_{D-1} = 1 - \frac{\beta}{1+b} [de^{-\beta x_{D-1}} x_D h_{D-1,D} + d^2 e^{-\beta(x_{D-1} + x_D)} h_{D-1,D+1}].$$

The constraints are $dh_{D-1,D} + d^2 h_{D-1,D+1} \leq (1+b)/\beta^2$. We claim that any $x_{D-1} \in [0, 1]$ is achievable here as well. To show that, put $h_{D-1,D} = 0$. The above equation becomes then,

$$\begin{aligned} x_{D-1} &= 1 - \frac{\beta}{1+b} d^2 e^{-\beta(x_{D-1} + x_D)} h_{D-1,D+1} \\ &= 1 - \frac{\beta}{1+b} d^2 e^{-\beta x_{D-1}} \frac{1+b}{d\beta h_{D,D+1}} h_{D-1,D+1}, \text{ from the earlier expression} \\ &= 1 - \frac{dh_{D-1,D+1}}{h_{D,D+1}} e^{-\beta x_{D-1}} \end{aligned}$$

This again can satisfy any x_{D-1} , since the coefficient of the exponential term can be made anywhere between 0 and 1. It can be made 0 by choosing $h_{D-1,D+1} = 0$, and 1 by choosing $\frac{dh_{D-1,D+1}}{h_{D,D+1}} = 1$ which is feasible, since $d^2 h_{D-1,D+1} = dh_{D,D+1} \leq (1+b)/\beta^2$.

In the similar way we can continue the induction till the root and can make $\mathbf{x}^* = \mathbf{x}^{\text{OPT}}$. Hence, PoA = 1.

Case 2 ($1 < \beta < \infty$): We note that this region of β falls in the region specified by Lemma 6. Hence the optimal effort is 1 for all the leaves and 0 for everyone else. Hence, the optimal social output is given by d^D . The equilibrium effort for the leaves, $x_{D+1} = 1$. However, Equation (24) may not be satisfiable for any x_D since $1/\beta < 1$. In order to push the solution as close to zero as possible, we choose $h_{D,D+1} = (1+b)/\beta^2$, and plug it in Equation (24), and the solution is given by $\xi(\beta)$ (recall Equation (9)) and the solution set is singleton under this condition. The social output is d^D , which is the numerator of the PoA expression. The denominator is given by the social output at the Nash equilibrium, which we will try to lower bound. From Equation (23), for the equilibrium, we know that $x_D = \xi(\beta)$. Therefore,

$$x_D + de^{-\beta x_D} = x_D + d\beta(1 - x_D) = d\beta + (1 - d\beta)\xi(\beta).$$

At the same time, we see that the leftmost expression is convex in x_D , which can be lower bounded by the minima, given by,

$$x_D + de^{-\beta x_D} \geq \frac{1}{\beta}(1 + \ln(d\beta)).$$

Combining the two, a tight lower bound of the expression would be,

$$x_D + de^{-\beta x_D} \geq \max \left\{ \frac{1}{\beta}(1 + \ln(d\beta)), d\beta + (1 - d\beta)\xi(\beta) \right\} = \phi(d, \beta).$$

Plugging this lower bound in Equation (23), we see that,

$$\begin{aligned} SO(\mathbf{x}) &\geq x_1 + de^{-\beta x_1}x_2 + \dots + d^{D-1}e^{-\beta(\sum_{i=1}^{D-1} x_i)} \cdot \phi(d, \beta) \\ &= x_1 + de^{-\beta x_1}x_2 + \dots + d^{D-2}e^{-\beta(\sum_{i=1}^{D-2} x_i)} \cdot (x_{D-1} + d\phi(d, \beta)e^{-\beta x_{D-1}}) \end{aligned}$$

Let us consider the last term within parenthesis.

$$\begin{aligned} x_{D-1} + d\phi(d, \beta)e^{-\beta x_{D-1}} &= x_{D-1} + d\phi(d, \beta)\frac{\beta}{\xi(\beta)}(1 - x_{D-1}) \\ &\geq x_{D-1} + d\phi(d, \beta)\beta(1 - x_{D-1}), \quad \text{as } \xi(\beta) \leq 1 \\ &= d\phi(d, \beta)\beta + (1 - d\phi(d, \beta)\beta)x_{D-1} \\ &\geq d\phi(d, \beta)\beta + (1 - d\phi(d, \beta)\beta)\xi(\beta) \end{aligned}$$

The first equality comes since we can make the equilibrium x_{D-1} s.t., $x_{D-1} = 1 - \frac{\xi(\beta)}{\beta}e^{-\beta x_{D-1}}$, by choosing $dh_{D-1,D} = (1+b)/\beta^2$, $d^2h_{D-1,D+1} = 0$. Also, since $\xi(\beta) \leq 1$, we conclude, $x_{D-1} \geq x_D = \xi(\beta)$, which gives the second inequality above. On the other hand, using the fact that the expression $x_{D-1} + d\phi(d, \beta)e^{-\beta x_{D-1}}$ is convex in x_{D-1} , it can be lower bounded by, $\frac{1}{\beta}(1 + \ln(d\phi(d, \beta)\beta))$. Combining this and the above inequality, we get the following.

$$\begin{aligned} SO(\mathbf{x}) &\geq x_1 + de^{-\beta x_1}x_2 + \dots + d^{D-2}e^{-\beta(\sum_{i=1}^{D-2} x_i)} \cdot \phi(d \cdot \phi(d, \beta), \beta) \\ &\quad \vdots \quad \text{repeating the steps above} \\ &\geq t_D(d, \beta), \quad \text{as defined in Equation (10)}. \end{aligned}$$

Therefore the PoA $\leq \frac{d^D}{t_D(d, \beta)}$. □

C Proofs for general networks

Proof of Lemma 3 Proof: We follow the line of proof of Theorem 1. Each agent $i \in N$ is solving the following optimization problem.

$$\begin{aligned} \max_{x_i} \quad & u_i(x_i, x_{-i}) \\ \text{s.t.} \quad & 0 \leq x_i \leq 1 \end{aligned} \tag{25}$$

This is a non-linear optimization problem. Hence we can write down the Lagrangian as follows.

$$\mathcal{L} = u_i(x_i, x_{-i}) + \lambda_i x_i + \gamma_i(1 - x_i), \quad \lambda_i, \gamma_i \geq 0.$$

The KKT conditions are necessary for this optimization problem (25), which are the following.

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial x_i} &= 0, \\ \Rightarrow \frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) + \lambda_i - \gamma_i &= 0, \end{aligned} \tag{26}$$

$$\lambda_i x_i = 0, \quad \gamma_i (1 - x_i) = 0. \tag{27}$$

Case 1: $\lambda_i = 0, \gamma_i = 0$, then from Equation (26) we get,

$$\begin{aligned} \frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) &= 0 \\ \Rightarrow p_i f'(x_i) + \sum_{j \in E_i} h_{ij} \frac{\partial p_j}{\partial x_i} x_j &= 0 \\ \Rightarrow f'(x_i) = \sum_{j \in E_i} h_{ij} \left(-\frac{1}{p_i} \frac{\partial p_j}{\partial x_i} x_j \right) &= g_i(\mathbf{x}) \\ \Rightarrow x_i = \ell \circ g_i(\mathbf{x}), \text{ from the definition of } \ell \end{aligned} \tag{28}$$

Case 2: $\lambda_i > 0, \gamma_i = 0$, then from Equation (27) we get $x_i = 0$, and from Equation (26),

$$\frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) < 0.$$

Carrying out the differentiation as in Equation (20), we get,

$$\begin{aligned} f'(x_i) < g_i(\mathbf{x}) &\Rightarrow 0 = x_i > \ell \circ g_i(\mathbf{x}), \text{ since } f \text{ is concave} \\ \Rightarrow x_i &= T \circ \ell \circ g_i(\mathbf{x}), \text{ where } T \text{ is the truncation function.} \end{aligned} \tag{29}$$

Case 3: $\lambda_i = 0, \gamma_i > 0$, then from Equation (27) we get $x_i = 1$, and from Equation (26),

$$\frac{\partial}{\partial x_i} u_i(x_i, x_{-i}) > 0.$$

Carrying out similar steps as before, we get,

$$\begin{aligned} f'(x_i) > g_i(\mathbf{x}) &\Rightarrow 1 = x_i < \ell \circ g_i(\mathbf{x}) \\ \Rightarrow x_i &= T \circ \ell \circ g_i(\mathbf{x}), \text{ where } T \text{ is the truncation function.} \end{aligned} \tag{30}$$

Case 4: $\lambda_i > 0, \gamma_i > 0$, this cannot happen since it will lead to a contradiction $0 = x_i = 1$. Therefore, combining Equations (28), (29), and (30), we get,

$$x_i^* = T \circ \ell \circ g_i(\mathbf{x}^*), \quad \forall i \in N.$$

Hence proved. □